Mathematics I — Problem Set 1

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Due: September 11, 2014 at 12.00. Remember to always use clear arguments, in the form of proofs or counter-examples (unless you are asked not to). If you find any typos, or think that something is unclear, please email me at adam@altmejd.se. Good luck!

Exercise 1 Drawing sets

Draw diagrams of the following sets in $\mathbb{R}^2$:

$$A = \left\{x : x_1^2 + x_2^2 \geq 1\right\} \cap [0,1]^2$$
$$B = \left\{x \in \mathbb{R}^2 : x_1 \in [0, x_2]\right\}$$
$$C = \{0,1,2,3\}^2$$

Exercise 2 Upper and lower bounds

Identify the (possibly empty) sets A and B in $\mathbb{R}$ of lower and upper bounds to the following sets (in $\mathbb{R}$):

(a) $\mathbb{Z}$

(b) $\mathbb{N}$

(c) $[0,1] \cup \{4,5\}$

(d) $]0,1[\cup\{4,5\}$

(e) $\{x \in \mathbb{R} : x = 1/n \text{ for some } n \in \mathbb{N}\}$.

Exercise 3 Supremum and infimum

Identify the supremum and infimum, if such exists, for each of the sets in Exercise 2.

Exercise 4 Proof of equivalence

Prove the following equivalence (where X is the universal set). Hint: De Morgan's laws can be useful! ($P^{\complement}$ is here the complement of the set P, i.e. $P^{\complement} = \{x \in X : x \notin P\}$)

$$P \subset Q \Leftrightarrow P^{\complement} \cup Q = X$$

Exercise 5 Bijection of a composite function (ex. 2.6.10)

Let $f : X \to Y$ and $g : Y \to Z$. Show that $g \circ f : X \to Z$ is a bijection if g and f are bijections.

Exercise 6 Correspondences (ex. 2.6.16)

Let $X = Y = \mathbb{R}^2_+$, and consider the correspondences φ and ψ from X to Y defined by $\varphi(x) = \{y \in \mathbb{R}^2 : y_1 \geq x_1 \text{ and } y_2 \geq x_2\}$ and $\psi(x) = \{y \in \mathbb{R}^2 : y_1 y_2 \geq x_1 x_2\}$. Draw pictures of the images of $x = (0,0)$, $x = (1,2)$ under φ and ψ. Note that ψ is the weak-preference correspondence of a consumer with Cobb-Douglas utility $u(x) = (x_1)^{1/2}(x_2)^{1/2}$; $\psi(x)$ is the set of consumption bundles y that he or she weakly prefers to x.

Exercise 7 Open and closed sets

Which of the following sets in $\mathbb{R}^2$ are open and/or closed? Find the closure and interior of each set. You do not need to provide formal proofs, short clear arguments are enough.

(a) $]0,1[\times]1,2[$

(b) $[0,1] \times [1,2]$

(c) $]0,1[\times [1,2]$

(d) a straight line through the origin

(e) its complement

(f) $]0,1[\times\{1,2,3\}$

(g) $[0,1] \times \{1,2,3\}$

Exercise 8 The open ball

Prove that $\forall x \in \mathbb{R}^n$ and any $\delta \in \mathbb{R}_{++}$, $B_\delta(x)$ is an open set.

Exercise 9 More set properties

For each of the following three sets, all in $\mathbb{R}^2$, decide which, if any, of the following properties each set has; finite, bounded, convex, open, closed, compact, connected. You do not have to provide formal proofs, drawing the sets and/or providing short comments is enough.

(a) $A =]1, 2[\times \{1, 2\}$

(b) $B = \left\{x \in \mathbb{R}^2_+ : x_1 \leq (2 - x_2)^2\right\}$

(c) $C = \cap_{x \in \mathbb{N}} \left\{x \in \mathbb{R}^2 : 0 < ||x|| < 1/n\right\}$

Exercise 10 Convexity under cartesian products

Suppose the sets $A_1, A_2 \subset \mathbb{R}$ are convex. Is then also $A = A_1 \times A_2$ convex?