Mathematics I — Problem Set 2

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Due: September 18, 2014 at 12.00. Hand in by either email or in my postbox, not both. You may use any theorems defined in the lecture notes as long as you refer to them. Any other statements must be proven or clearly shown (unless stated otherwise). You are free to collaborate, but please hand in your own solutions, do not just copy your friends'.

If you find any typos, or think that something is unclear, please email me at adam@altmejd.se. Good luck!

Exercise 1 Cobb-Douglas (ex. 3.15)

Let $\alpha \in]0,1[$ and let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x) = x_1^\alpha x_2^{1-\alpha}$ for all $x \in \mathbb{R}^2_+$, otherwise $f(x) = 0$.

(a) Draw a contour map of f.

(b) Is f quasi-concave?

(c) Is f concave?

Exercise 2 Concavity under composition

Prove the following theorem.

Theorem 1

Let $f : \mathbb{R}^n \supset X \to \mathbb{R}$ be a concave function and $g : \mathbb{R} \supset I \to \mathbb{R}$ and increasing and concave function such that $f(X) \subset I$. Then the function $g \circ f$ is concave.

Exercise 3 Compactness

Which of the following sets are compact?

(a) $A = \{x \in \mathbb{N} : 0 \leq x \leq 10\}$

(b) $B = \{x \in \mathbb{R} : 0 < x \leq 2\}$

(c) $C = A \cup B$

(d) $D = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$

(e) $E = \mathbb{Q} \cap [0, 1] \subset \mathbb{R}$

(f) $F = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

(g) $G = F \cup \{0\}$

Exercise 4 Compactness from definition (ex. 4.19)

Prove directly from the definition that the set $X =]0, 1[$ is not compact. (Hint: use a covering like $\mathcal{C}$ consisting of the sets $C_k = \left]\frac{1}{k+5}, \frac{k}{k+1}\right[$, for all $k \in \mathbb{N}$.

Exercise 5 Properties under continuous maps (ex. 5.6)

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\emptyset \neq X \subset \mathbb{R}^2$. Which of the following claims are true?

(a) If X is open, then so is $f(X)$.

(b) If X is bounded, then

$$X^* = \arg\max_{x \in X} f(x) = \left\{x^* \in \mathbb{R}^2 : f(x^*) \geq f(x) \forall x \in X\right\}$$

is non-empty

(c) If X is closed, then X^* is non-empty.

(d) If X is compact, then X^* is non-empty.

Exercise 6 L.H.C (ex. 5.10)

Show that the demand correspondence in Example 63 on page 90 in the lecture notes is *not* l.h.c. at $p = 1$. Verify that, by contrast, the indirect utility function is indeed continuous.

Exercise 7 Proof of continuity

Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if x is irrational} \\ 1/q & \text{if x is the rational number p/q} \end{cases}$$

where p, q are integers with no common factor and $q > 0$. Show that f is discontinuous at all rational numbers an continuous at all irrational ones.

Exercise 8 Brouwer's

For each of the following sets $X \subset \mathbb{R}$, find a continuous function $f : X \to X$, such that $f(X) \subset X$. For (a) and (b) your function should have no fixed point. For (c) and (d) it should have at least one. Relate Brouwer's Fixed Point Theorem to your examples.

(a) $X = [-1, 0[$

(b) $X = B_2(0) \setminus \{0\}$

(c) $X = [0, 1]$

(d) $X = \{1, 2, 3, 4, 5\}$

Exercise 9 Kakutani's (ex. 6.8)

Verify whether or not the Kakutani Fixed-Point Theorem applies to the following two correspondences from the unit square $X = [0, 1]^2$ to itself. Draw pictures to illustrate each correspondence. For each correspondence, explain which conditions hold and fail (no formal proofs needed), and identify the set of fixed points.

(a) $\varphi(x) = \left\{y \in X : \|y - x\| \leq \frac{1}{5}\right\}$

(b) $\psi(x) = \left\{y \in X : \|y - x\| \geq \frac{1}{10}\right\}$

(c) $\xi(x) = \arg\max_{y \in X} \|y - x\|$

(d) $\gamma(x) = \mathrm{co}\,[\xi(x)]$