Mathematics I — Problem Set 3

Adam Altmejd

2014-09-19

Due: September 25, 2014 at 12.00. Hand in by either email or in my postbox, not both. You may use any theorems defined in the lecture notes as long as you refer to them. Any other statements must be proven or clearly shown (unless stated otherwise).

You are free to collaborate, but please hand in your own solutions, do not just copy your friends', and specify clearly in the header who you are working with.

If you find any typos, or think that something is unclear, please email me at adam@altmejd.se. Good luck!

Exercise 1 Sequences

Consider the following sequences from $X = \mathbb{R}$, defined for each $t \in \mathbb{N}$ as follows:

(a) $x^t = t$

(b) $y^t = 2t$

(c) $z^t = 1/t$

(d) $v^t = (-1)^t$

(e) $w^t = \frac{1-t}{1+t}$

For each sequence, identify its range and decide whether it is convergent or divergent. If possible, identify subsequences that converge to distinct points. Decide which, if any, of the sequences are Cauchy.

Exercise 2 Bernoulli's inequality and sequences

Consider the following:

Proposition 1: Bernoulli's inequality

for each $n \in \mathbb{N}$ and $x \geq -1$,

$$(1+x)^n \geq 1+nx$$

First, prove Bernoulli's inequality by induction, then use it to prove the following:

Let $\alpha \in \mathbb{R}$ and consider the sequence $(x^t)_{t\in\mathbb{N}}$ defined as $x^t = \alpha^t$. Then:

(a) if $|\alpha| < 1$, $(x^t) \to 0$,

(b) if $\alpha > 1$, $(x^t) \to \infty$.

(c) if $\alpha < -1$, (x^t) diverges.

Exercise 3 Continuity with sequences

Given a function $f : \mathbb{R} \to \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(x) = (x, f(x))$. Use the sequential characterization of continuity to show that if f is continuous at some point x^0 so is g.

Exercise 4 Contraction mapping

Show that the function $f(x) = \sqrt{x}$ defines a contraction on $[1, \infty[$. Approximate its fixed point using successive iterations of the function (if you know how, write a program to do this for you, just doing it manually or in e.g. excel works great too). Show in a graph what is going on.

Exercise 5 Properties of solution set (exam 2013)

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are *continuous* and let $X^* = \arg\max_{x\in X} f(x)$, where $X = \left\{x \in \mathbb{R}^2 : g(x) \geq 0\right\}$. Which of the following claims are true? Give a proof or a counter-example.

(a) If X is *bounded*, then X^* is non-empty.

(b) If f and g are *quasi-concave*, then X^* is convex.

(c) If f is differentiable, $g\left(x^0\right) > 0$ and $\nabla f\left(x^0\right) = \mathbf{0}$, then $x^0 \in X^*$.

Exercise 6 Closedness and upper hemi-continuity

Using the sequential defintions of upper hemi-continuity to prove a slight variation of the closed graph proposition:

Proposition 1

Suppose $\varphi : X \rightrightarrows Y$ has a closed graph. If for any bounded set B in X, $\varphi(B)$ is bounded, then φ is u.h.c.

You will need the following theorem

Theorem 2

A sequence (x^t) from $\mathbb{R}^n$ converges to a vector $x = (x_1, x_2, \ldots, x_n)$ if and only if each coordinate sequence (x_i^t) converges to x_i.

Exercise 7 Continuity of the constraint (ex 5.11)

The following examples show that neither upper nor lower hemi-continuity of the constraint correspondence γ is sufficient for the solution correspondence ξ of $[M]$ to be *u.h.c.*:

(a) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$, and let the correspondence $\gamma : \mathbb{R} \rightrightarrows [0,1]$ be defined by $\gamma(a) = [0,1]$ for $a \leq 2$ and $\gamma(a) = \left\{\frac{1}{2}\right\}$ for $a > 2$. Verify that γ is compact-valued and *u.h.c.*, that the solution correspondence is not u.h.c. and that the value function v is not continuous.

(b) Modify the constraint correspondence γ in (a) so that $\gamma(2) = \left\{\frac{1}{2}\right\}$. What happens to the continuity properties of γ, ξ and v?

Exercise 8 Kakutani again (exam 2013)

(a) Consider the correspondence $\varphi : X \rightrightarrows X$, wehere $X = [0,1]^2$, defined by

$$\varphi(x,y) = \left[\arg\max_{x' \in [0,1]} u\left(x', y\right)\right] \times \left[\arg\max_{y' \in [0,1]} v\left(x, y'\right)\right]$$

where u and v are *continuous* and *concave* functions. Does φ have a fixed point? (For each hypothesis in Kakutani's theorem, verify whether it is met or not, and motivate your answers.)

(b) Let $u\left(x,y\right) \equiv 3x - y$ and $v\left(x,y\right) \equiv x - \left(x-y\right)^2$. Identify the (potentially empty) set of fixed points of the correspondence φ defined in (a).

Exercise 9 Cobb-Douglas

Consider the standard Cobb-Douglas production function with constant returns to scale; $f : \mathbb{R}^2_+ \to \mathbb{R}$ with $f(L,K) = L^\alpha K^{1-\alpha}$ and $\alpha \in]0,1[$. At which points in its domain is f (a) continuous, and (b) differentiable?

Exercise 10 Differentiability

Given the following utility function:

$$u(c) = c$$

Consider a consumer who has such a utility function that is multiplicative over two goods ($c = (c_1, c_2)$), $U : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$U(c) = \begin{cases} u(c_1)u(c_2) & c \in \mathbb{R}^2_+ \\ 0 & \text{otherwise} \end{cases}$$

First try to show that the function is differentiable at $(0, 0)$ using Proposition 41 from the lecture notes. If this does not work, use the definition to show that the function is in fact differentiable. (Hint: $c_1 c_2 = \|c\|^2 \sin(\alpha)\cos(\alpha)$ with α such that $\tan(\alpha) = c_2/c_1$.)