Mathematics I — Problem Set 4

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Due: October 2, 2014 at 12.00. Hand in by either email or in my postbox, not both. You may use any theorems defined in the lecture notes as long as you refer to them. Any other statements must be proven or clearly shown (unless stated otherwise).

You are free to collaborate, but please hand in your own solutions, do not just copy your friends', and specify clearly in the header who you are working with.

If you find any typos, or think that something is unclear, please email me at adam@altmejd.se. Good luck!

Exercise 1 Directional Derivative

Calculate the directional derivatives of the following functions at the given points and directions.

(a) $f(x, y) = 2x + y - 1$ at $x^0 = (2, 1)$ in the direction given by a vector $(1, 1)$ starting in the point x^0.

(b) $f(x, y, z) = xe^{xy} - xy - z^2$ at $x^0 = (0, 1, 1)$ in the direction given by $(1, 1, 1)$ starting in x^0.

Exercise 2 Lipschitz continuity

For each of the three functions $f : \mathbb{R} \to \mathbb{R}$ defined below, decide if it is (i) continuous, (ii) locally Lipschitz continuous, (iii) globally Lipschitz continuous, (iv) concave, and/or (v) quasi-concave.

(a) $f(x) = x^2$

(b) $f(x) = |x|$

(c) $f(x) = |x|^{1/2}$

Exercise 3 Maximization (almost ex 8.9)

Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by $f(x) = x_1 x_2$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, a) = 1 - x_1/a - x_2$, where $a > 0$.

(a) Draw a diagram of the set $X(a) = \left\{x \in \mathbb{R}^2_+ : g(x, a) \geq 0\right\}$ and indicate the level curves $f(x) = 1$ and $f(x) = 2$. Draw the gradient of f as an arrow in $\mathbb{R}^2$ at points $x^0 = (1, 1)$, $x^1 = (1, 2)$, $x^2 = (2, 1)$. Also, find and draw the corresponding derivatives (affine approximations) $h^0(x)$, $h^1(x)$, $h^2(x)$.

(b) Give arguments for the existence of a solution, and verify that no solution has $x_1 = 0$ or $x_2 = 0$.

(c) Identify the boundary point $x^*(a)$ that, for a fixed and given, meets the necessary first-order condition $\nabla f(x^*) + \lambda \nabla g(x^*, a) = \mathbf{0}$ for a solution on the boundary.

(d) Give arguments that show (or at least suggest) that the point $x^*(a)$ indeed is the solution.

(e) Find the maximum value $v(a)$ of f over $X(a)$ and draw a diagram of the graph of the function v.

Exercise 4 Local uniqueness (ex. 9.3)

Is the solution $x^0 = (1, 1, 1)$ to the following system of non-linear equations locally unique?

$$\begin{cases} x_1^2 + 2x_2^2 + 4x_3^2 = 7 \\ 4x_1 x_2^2 + 7x_2 x_3 = 11 \\ \log(x_1 x_2 x_3) = 0 \end{cases}$$

Exercise 5 System of equations (ex. 9.4)

Consider the following system of equations in $x \in \mathbb{R}^3_+$, for a given parameter $a \in \mathbb{R}$:

$$\begin{cases} x_1 + 2\sqrt{x_2}\, x_3 = 3 \\ x_2 x_3 + x_1 x_2 = 2 \\ 2x_1 x_2 x_3 + 4 = a \end{cases}$$

(a) Write the equation system in the form $f(x, a) = \mathbf{0}$, where $f : \mathbb{R}^4_+ \to \mathbb{R}^3$ and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^3$.

(b) For what value a^0 of the parameter a is $x^0 = (1, 1, 1)$ a solution?

(c) Is f continuously differentiable at x^0?

(d) Suppose $a = a^0$. Is the solution $x^0 = (1, 1, 1)$ then locally unique?

(e) Does the equation system define the solution $x \in \mathbb{R}^3$ as an implicitly defined function g of a, $x = g(a)$, for values of a near a^0?

Exercise 6 Maximization with KKT (ex. 9.12)

Consider the program $\max(x_1 + x_2)$, subject to the constraints $x_1^2 + x_2 \leq 2$ and $x_1 \leq a$ where a is a positive scalar.

(a) Find constraint functions g_1 and g_2 for each of the two constraints, and write the program in such a form that the Kuhn-Tucker Theorem can be used.

(b) Identify the set X^{CQ} of points x that meet the constraint qualification.

(c) Use the Kuhn-Tucker Theorem to identify the solution set for all $a > 0$. (Hint: there are 2 cases)

(d) How does the solution set change when a changes? Is the solution correspondence u.h.c.? Is it l.h.c.?

(e) How does the maximum value change? Is it continuous in a?

Exercise 7 Uncountable set of sequences

Note: This exercise is voluntary for those of you would like to do some more practice on sequences. I will discuss it in class.

Let $S = \{(x^t) : x_i \in \{0, 1\}\}$, i.e. the set of all sequences containing only 0 and 1. Show that S is uncountable.