Mathematics I — Problem Set 5

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Due: October 10, 2014 at 12.00. Hand in by either email or in my postbox, not both. You may use any theorems defined in the lecture notes as long as you refer to them. Any other statements must be proven or clearly shown (unless stated otherwise).

You are free to collaborate, but please hand in your own solutions, do not just copy your friends', and specify clearly in the header who you are working with.

If you find any typos, or think that something is unclear, please email me at adam@altmejd.se. Good luck!

Exercise 1 Directional Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Calculate $\nabla f(x,y)$

(b) Use the following alternative definition of a directional derivative

> **Definition 1: Directional derivative**
>
> Let $\mathbf{u} = u\mathbf{i} + u\mathbf{j}$ be a unit vector, so that $\|u\| = \sqrt{u^2+v^2} = 1$. The directional derivative of $f(x,y)$ at (a,b) in the direction of $\mathbf{u}$ is the rate of change of $f(x,y)$ with respect to distance measured at (a,b) along a ray in the direction of $\mathbf{u}$ in the xy-plane. Given by:
>
> $$D_{\mathbf{u}}f(a,b) = \lim_{h\to 0} \frac{f(a+hu, b+hv) - f(a,b)}{h}$$

to calculate $D_{\mathbf{u}}f(0,0)$, that is the directional derivative of f in the point $(0,0)$ in direction $\mathbf{u} = (\mathbf{i}+\mathbf{j})/\sqrt{2}$ with $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ being the unit vectors.

(c) Is $f(x,y)$ differentiable at $(0,0)$? Why or why not?

Exercise 2 KKT (exam 2013)

Consider the problem $\max_{x \in B} f(x,a)$ where $B = \left\{ x \in \mathbb{R}^2_+ : x_1 \leq (2 - x_2)^3 \right\}$, and $f(x,a) \equiv a x_1^2 + x_2$ for some scalar $a > 0$.

(a) Does the maximization program have at least one solution? (You may refer to precisely stated theorems.)

(b) Does f have a *gradient* at each point in B? Find the gradient wherever defined.

(c) Formulate the problem in such a way that you can use the Kuhn-Tucker theorem, and state precisely what the theorem says.

(d) Identify the points $x \in B$ that satisfy the *constraint qualification* and, using the Kuhn-Tucker theorem, solve the program for all values of $a > 0$.

Exercise 3 The Implicit-Function Theorem (ex 9.6)

Consider the continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,a) = x^2 + a^2 - 1$. What does the Implicit-Function Theorem say about solution function, when (a) $a^0 = 0$, when (b) $a^0 = 1/2$, and when (c) $a^0 = 1$? Draw a diagram showing the set where $f(x,a) = 0$.

Exercise 4 Another KKT (ex 9.14)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 2x + y$, and let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be given by $g_1(x,y) = -y$ and $g_2(x,y) = y - (x-1)^3$. Draw a picture of the set X, identify the subsets X^{CQ}, X^* and $\tilde{X}$.

Exercise 5 Intertemporal Optimization (example 122)

Consider the program:

$$\begin{aligned} &\max_{x \in X} u(x,\delta) \\ \text{s.t. } & u(x,\delta) = (x_1+1)^{\frac{1}{2}} + \delta (x_2+1)^{\frac{1}{2}} && \delta \in]0,1[\\ & X(p,w) = \left\{ x \in \mathbb{R}^2_+ : p \cdot x \leq w \right\} && p \in \mathbb{R}^2_{++}, w \in \mathbb{R}_{++} \end{aligned}$$

(a) Draw a diagram showing the set X and some isoquants of u for $p_1 = 2$, $p_2 = 1$, $w = 4$ and $\delta = 0.5$.

(b) Draw another diagram of $X(p,w)$ and indicate the gradients $\nabla u(x)$ and $\nabla g_i(x)$ and some points $x \in X$. Specify the set $\tilde{X}$.

(c) Does the optimization problem have a solution for all $p \in \mathbb{R}^2_{++}$, $w \in \mathbb{R}_{++}$ and $\delta \in]0,1[$?

(d) Use the Kuhn-Tucker Theorem find the solution to the program for any $p \in \mathbb{R}^2_{++}$ and $w \in \mathbb{R}_{++}$ and $\delta \in (0,1)$.

(e) What does Berge's theorem say about this problem?

Exercise 6 Sequence of averages

Note: This exercise is voluntary.

Show that for every convergent sequence $(s^t)_{t\in\mathbb{N}} \to s$ there exists a sequence of averages

$$\bar{s}^t = \frac{s_1 + s_2 + \ldots + s_t}{t}$$

s.t. $(\bar{s}^t)_{t\in\mathbb{N}} \to s$

Give an example of a sequence that is divergent but has a convergent sequence of averages.