Mathematics I — Problem Set 1

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Due: September 11, 2015 at 12.00. Hand in by email or in my mailbox.

Remember to always use clear arguments, using proofs and counter-examples when needed. Please write clearly (computer typed solutions are greatly appreciated) and carefully state definitions and theorems from the Lecture Notes whenever you use them. Collaboration in smaller groups is encouraged, however everyone needs to hand in their own solutions.

Grading is pass/fail. There will be 5 problem sets in total, each having equal weight. A pass will be rewarded as long you show that you understand and have made an honest effort on all questions.

Exercise 1 Drawing sets

Draw diagrams of the following sets in $\mathbb{R}^2$:

(a) $A = \{x \in \mathbb{R}^2 : 0 < x_1^2 + x_2^2 < 1\}$

(b) $B = \{x \in \mathbb{R}^2 : x_1 + x_2 = z \text{ for some } z \in \mathbb{Z}\}$

(c) $C = \mathbb{Z}^2 \setminus \{x \in \mathbb{R}^2 : |x_1| = |x_2|\}$

Exercise 2 Sets

Prove that

$$\bigcap_{i \in \mathbb{N}} A_i \cup \bigcap_{i \in \mathbb{N}} B_i \subset \bigcap_{i \in \mathbb{N}} (A_i \cup B_i)$$

and that equality does not have to hold.

*If you find any typos, or think that something is unclear, please email me at adam.altmejd@phdstudent.hhs.se. Good luck!

Exercise 3 Upper and lower bounds

Identify the (possibly empty) sets A and B of lower and upper bounds respectively of the following sets (in $\mathbb{R}$). Also specify the supremum and infimum of each set.

(a) $\mathbb{Z}$

(b) $\mathbb{R}_{++}$

(c) $\{x \in \mathbb{R} : x = n^2 \text{ for some } n \in \mathbb{N}\}$

(d) $(1, 2) \cup \{1, 2, 3, 4\}$

(e) $\bigcup_{n \in \mathbb{N}} \frac{n-1}{n}$

Exercise 4 Functions (ex. 2.6.11)

Let $f : X \to Y$ and $A \subset X$. Prove that $A \subset f^{-1}(f(A))$. Prove that this inclusion is an equality if f is injective.

Exercise 5 Correspondences (ex. 2.6.15)

Let $X = Y = [0, 1]$, and consider the correspondence $\phi : X \rightrightarrows Y$ defined by $\phi(x) = [1/4, 3/4]$ $\forall x \leq 1/2$ and $\phi(x) = x^2$ $\forall x > 1/2$. Draw a picture of its graph. Find two selections from ϕ.

Exercise 6 Correspondences 2 (ex. 2.6.16)

Let $X = Y = \mathbb{R}^2_+$, and consider the correspondences φ and ψ from X to Y defined by $\varphi(x) = \{y \in \mathbb{R}^2 : y_1 \geq x_1 \text{ and } y_2 \geq x_2\}$ and $\psi(x) = \{y \in \mathbb{R}^2 : y_1 y_2 \geq x_1 x_2\}$. Draw pictures of the images of $x = (0, 0)$, $x = (1, 2)$ under φ and ψ. Note that ψ is the weak-preference correspondence of a consumer with Cobb-Douglas utility $u(x) = (x_1)^{1/2}(x_2)^{1/2}$; $\psi(x)$ is the set of consumption bundles y that he or she weakly prefers to x.

Exercise 7 Unit discs (ex. 3.6.2 (e))

Draw a picture of the unit disc $D(0, 1) = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ in $\mathbb{R}^2$, where $|| \cdot || : \mathbb{R}^2 \to \mathbb{R}_+$ is each of the following norms:

(a) $||x||_1 = \sum_{i=1}^{2} |x_i|$

(b) $||x||_2 = \left(\sum_{i=1}^{2} |x_i|^2\right)^{\frac{1}{2}}$

(c) $||x||_\infty = \max_{i \in \{1,2\}} |x_i|$

Exercise 8 Open and closed sets

Which of the following sets are open and/or closed? Find the closure, interior and convex hull of each set. You do not need to provide formal proofs, short clear arguments are enough.

(a) $(0,1)^2 \in \mathbb{R}^2$

(b) $(0,1) \times [0,1]$

(c) $\{x, y\}$ for some $x \neq y \in \mathbb{R}^n$

(d) a hyperplane through the origin in $\mathbb{R}^n$

(e) its complement

(f) $]0,1[\times\{1,2,3\} \in \mathbb{R}^2$

(g) $\{1,2,3\}^3 \in \mathbb{R}^3$

Exercise 9 Union of closed sets

Give an example of an infinite union of closed sets that is not closed, and an infinite intersection of open sets that is not open (that are different from the examples in the notes).

Exercise 10 Set examples (ex 4.7.6)

Give an example (by defining each set properly and sketching it in a plane) of a non-empty proper subset of $\mathbb{R}^2$ which is:

(a) neither open nor closed

(b) non-convex and unbounded

(c) infinite and bounded with empty interior

(d) non-convex with convex closure

(e) not closed with open closure

make sure to motivate why your set has the desired properties.

Exercise 11 Uncountable power set (ex. 2.6.8)

Prove that $\mathcal{P}(\mathbb{N})$ is uncountable. (Hint: Prove first that if A is an arbitrary set, then there exist no surjection $g : A \to \mathcal{P}(A)$)