Mathematics I — Problem Set 2

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Due: September 17, 2015 at 12.00. Hand in by email or in my mailbox.

Remember to always use clear arguments, using proofs and counter-examples when needed. Please write clearly (computer typed solutions are greatly appreciated) and carefully state definitions and theorems from the Lecture Notes whenever you use them. Collaboration in smaller groups is encouraged, however everyone needs to hand in their own solutions.

Grading is pass/fail. There will be 5 problem sets in total, each having equal weight. A pass will be rewarded as long you show that you understand and have made an honest effort on all questions.

Exercise 1 Set properties

Which of the following statements are true for all subsets of S and T of $\mathbb{R}^n$?

(a) $\operatorname{int}(\bar{S}) = \operatorname{int}(S)$

(b) $\overline{S \cup T} = \bar{S} \cup \bar{T}$

(c) $\partial S \subset S$

(d) S is open $\Rightarrow S \cap \bar{T} \subset \overline{S \cap T}$

Exercise 2 Concave functions (ex. 3.5.16)

Give your own examples of functions $f : \mathbb{R}^2 \to \mathbb{R}$ which are:

(a) quasi-concave but not concave

(b) quasi-concave but not strictly quasi-concave

(c) concave but not strictly concave

*If you find any typos, or think that something is unclear, please email me at adam.altmejd@phdstudent.hhs.se. Good luck!

Exercise 3 Compactness

Which of the following sets are compact?

(a) $A = \{x \in \mathbb{N} : 0 \leq x \leq 10\} \subset \mathbb{R}$

(b) $B = \{x \in \mathbb{R} : 0 < x \leq 2\} \subset \mathbb{R}$

(c) $C = A \cup B \subset \mathbb{R}$

(d) $D = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$

(e) $E = \mathbb{Q} \cap [0, 1] \subset \mathbb{R}$

(f) $F = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

(g) $G = F \cup \{0\}$

Exercise 4 Some Topology

(a) Let $A \subset \mathbb{R}^n$ be an open set and $B \subset \mathbb{R}^n$ be any subset of $\mathbb{R}^n$. Define $A + B \equiv \{a + b \in \mathbb{R}^n : a \in A \wedge b \in B\}$. Prove that $A + B$ is open.

(b) Let $\mathbb{Q}$ be the set of all rational numbers. Find $int(\mathbb{Q})$, $\partial(\mathbb{Q})$ and $\bar{\mathbb{Q}}$. Let $S = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. Is S closed?

(c) Let $A_1, A_2, \ldots \subset \mathbb{R}^n$ be compact sets. Prove that $\bigcap_{i \in \mathbb{N}} A_i$ is a compact set.

(d) Let $A \subset \mathbb{R}^n$ be dense in $\mathbb{R}^n$. Then, show that the only open set in $\mathbb{R}^n$ which is disjoint from A is $\emptyset$.

(e) Prove directly from the definition that the set $X = (0, 1)$ is not compact. (Hint: use a covering like $\mathcal{C}$ consisting of the sets $C_k = \left(\frac{1}{k+5}, \frac{k}{k+1}\right)$, for all $k \in \mathbb{N}$.) (ex. 4.7.19)

Exercise 5 Continuity (ex. 5.5.3)

Verify that the (projection) function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = x$, is continuous. Find a closed set $C \subset \mathbb{R}^2$ such that $f(C) \subset \mathbb{R}$ is not closed.

Exercise 6 Properties under continuous mappings (ex. 5.5.6)

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\emptyset \neq X \subset \mathbb{R}^2$. Which of the following claims are true?

(a) If X is open, then so is $f(X)$.

(b) If X is bounded, then

$$X^* = \arg\max_{x \in X} f(x) = \left\{x^* \in \mathbb{R}^2 : f(x^*) \geq f(x) \forall x \in X\right\}$$

is non-empty

(c) If X is closed, then X^* is non-empty.

(d) If X is compact, then X^* is non-empty.

Exercise 7 Continuity of a correspondence

Let $\varphi : X \rightrightarrows Y$ for $X = Y = [0,1]$ be defined by

$$\varphi(x) = \begin{cases} \left[0, \frac{1}{2}\right] & x \leq \frac{1}{2} \\ \{x^2\} & x > \frac{1}{2} \end{cases}$$

(a) Draw the graph of φ

(b) Is φ upper hemi-continuous?

(c) Is φ lower hemi-continuous?

(d) Modify the correspondence as follows: $\psi : X \rightrightarrows Y$ defined by

$$\psi(x) = \begin{cases} \left[0, \frac{1}{2}\right] & x < \frac{1}{2} \\ \{x^2\} & x \geq \frac{1}{2} \end{cases}$$

Draw graph(ψ). Is ψ upper hemi-continuous? lower hemi-continuous?

Exercise 8 Brouwer's

For each of the following sets $X \subset \mathbb{R}$, find a continuous function $f : X \to X$, such that $f(X) \subset X$. For (a) and (b) your function should have no fixed point. For (c) and (d) it should have at least one. Relate Brouwer's Fixed Point Theorem to your examples.

(a) $X = [-1, 0)$

(b) $X = B_2(0) \setminus \{0\}$

(c) $X = [0, 1]$

(d) $X = \{1, 2, 3, 4, 5\}$