Mathematics I — Problem Set 3

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Due: September 24, 2015 at 12.00. Hand in by email or in my mailbox.

Remember to always use clear arguments, using proofs and counter-examples when needed. Please write clearly (computer typed solutions are greatly appreciated) and carefully state definitions and theorems from the Lecture Notes whenever you use them. Collaboration in smaller groups is encouraged, however everyone needs to hand in their own solutions.

Grading is pass/fail. There will be 5 problem sets in total, each having equal weight. A pass will be rewarded as long you show that you have made an honest effort on all questions.

If there is something that you find difficult, or have trouble understanding please write so, instead of just writing down a solution that you did not comprehend.

Exercise 1 Sequences

Consider the following sequences from $X = \mathbb{R}$, defined for each $t \in \mathbb{N}$ as follows. For each sequence, identify its range and decide whether it is convergent or divergent. If possible, identify subsequences that converge to distinct points. Decide which, if any, of the sequences are Cauchy.

(a) $x^t = t$

(b) $y^t = 2t$

(c) $z^t = 1/t$

(d) $v^t = (-1)^t$

(e) $w^t = \frac{1-t}{1+t}$

Exercise 2 Continuity with sequences

Given a function $f : \mathbb{R} \to \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(x) = (x, f(x))$. Use the sequential characterization of continuity to show that if f is continuous at some point x^0 so is g.

*If you find any typos, or think that something is unclear, please email me at adam.altmejd@phdstudent.hhs.se. Good luck!

Exercise 3 Contraction mapping

Show that the function $f(x) = \sqrt{x}$ defines a contraction on $[1, \infty)$. Approximate its fixed point using successive iterations of the function (you can do this in excel or by writing a simple program). Show in a graph what is going on.

Exercise 4 Concave function (ex. 3.5.13)

Show that a function $f : \mathbb{R} \to \mathbb{R}$ is concave if and only if its subgraph is convex.

Exercise 5 Properties of solution set (exam 2013)

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are continuous and let $X^* = \arg\max_{x \in X} f(x)$, where $X = \left\{x \in \mathbb{R}^2 : g(x) \geq 0\right\}$. Which of the following claims are true? Give a proof or a counter-example.

(a) If X is *bounded*, then X^* is non-empty.

(b) If f and g are *quasi-concave*, then X^* is convex.

(c) If f is differentiable, $g\left(x^0\right) > 0$ and $\nabla f\left(x^0\right) = \mathbf{0}$, then $x^0 \in X^*$.

Exercise 6 Kakutani 1 (ex. 6.4.5)

Let $X = [0, 1]$ and show that the correspondence $\varphi : X \rightrightarrows X$, defined by $\varphi(x) = \left\{\frac{x}{2} + \frac{1}{4}\right\}$ for $x < \frac{1}{2}$ and $\varphi(x) = \left[0, \frac{1}{2}\right)$ for $x \geq \frac{1}{2}$ is u.h.c. but has no fixed point. Which hypothesis of the Kakutani's Fixed Point Theorem is violated?

Exercise 7 Kakutani 2

Verify whether or not the Kakutani Fixed-Point Theorem applies to the following four correspondences from the unit square $X = [0, 1]^2$ to itself. Draw pictures to illustrate each correspondence. For each correspondence, explain which conditions hold and fail, and identify the set of fixed points.

(a) $\varphi(x) = \left\{y \in X : \|y - x\| \leq \frac{1}{5}\right\}$

(b) $\psi(x) = \left\{y \in X : \|y - x\| \geq \frac{1}{10}\right\}$

(c) $\xi(x) = \arg\max_{y \in X} \|y - x\|$

(d) $\gamma(x) = \mathrm{co}\left[\xi(x)\right]$

Exercise 8 Question 9.6:

Depict in a diagram the Leontief function $f : \mathbb{R}^2_+ \to \mathbb{R}$, defined by $f(x) = \min\{a_1 x_1, a_2 x_2\}$ for parameters $a_1 = 1$ and $a_2 = 2$. Study in particular the function's behavior along the straight line L defined by the equation $x_1 = 2x_2$.

Verify grafically that f is continuous, that it is differentiable at all points $x \notin L$. Explain in words why it is not differentiable on L. Indicate in a diagram the subsets of $\mathbb{R}^2_+$ on which f takes the values 1,2 and 3, respectively.

Exercise 9 Cobb-Douglas

Consider the standard Cobb-Douglas production function with constant returns to scale; $f : \mathbb{R}^2_+ \to \mathbb{R}$ with $f(L, K) = L^\alpha K^{1-\alpha}$ and $\alpha \in (0, 1)$. At which points in its domain is f (a) continuous, and (b) differentiable?