Mathematics I — Problem Set 4

Adam Altmejd*

2015–09–25

Due: October 1, 2015 at 12.00. Hand in by email or in my mailbox.

Remember to always use clear arguments, using proofs and counter-examples when needed. Please write clearly (computer typed solutions are greatly appreciated) and carefully state definitions and theorems from the Lecture Notes whenever you use them. Collaboration in smaller groups is encouraged, however everyone needs to hand in their own solutions.

Grading is pass/fail. There will be 5 problem sets in total, each having equal weight. A pass will be rewarded as long you show that you have made an honest effort on all questions.

If there is something that you find difficult, or have trouble understanding please write so, instead of just writing down a solution that you did not comprehend.

Exercise 1 Continuity of the constraints (ex 5.11)

The following examples show that neither upper nor lower hemi-continuity of the constraint correspondence γ is sufficient for the solution correspondence ξ to be *u.h.c.*:

(a) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$, and let the correspondence $\gamma : \mathbb{R} \rightrightarrows [0,1]$ be defined by $\gamma(a) = [0,1]$ for $a \leq 2$ and $\gamma(a) = \left\{\frac{1}{2}\right\}$ for $a > 2$. Verify that γ is compact-valued and *u.h.c.*, that the solution correspondence ξ is not u.h.c. and that the value function v is not continuous.

(b) Modify the constraint correspondence γ in (a) so that $\gamma(2) = \left\{\frac{1}{2}\right\}$. What happens to the continuity properties of γ, ξ and v?

*If you find any typos, or think that something is unclear, please email me at adam.altmejd@phdstudent.hhs.se. Good luck!

Exercise 2 Kakutani (exam 2013)

(a) Consider the correspondence $\varphi : X \rightrightarrows X$, wehere $X = [0,1]^2$, defined by

$$\varphi(x,y) = \left[\arg\max_{x' \in [0,1]} u\left(x', y\right)\right] \times \left[\arg\max_{y' \in [0,1]} v\left(x, y'\right)\right]$$

where u and v are *continuous* and *concave* functions. Does φ have a fixed point? (For each hypothesis in Kakutani's theorem, verify whether it is met or not, and motivate your answers.)

(b) Let $u\left(x,y\right) \equiv 3x - y$ and $v\left(x,y\right) \equiv x - \left(x-y\right)^2$. Identify the (potentially empty) set of fixed points of the correspondence φ defined in (a).

Exercise 3 Maximization (almost ex 8.9)

Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by $f(x) = x_1 x_2$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be by $g(x,a) = 1 - x_1/a - x_2$, where $a > 0$.

(a) Draw a diagram of the set $X(a) = \left\{x \in \mathbb{R}^2_+ : g(x,a) \geq 0\right\}$ and indicate the level curves $f(x) = 1$ and $f(x) = 2$. Draw the gradient of f as an arrow in $\mathbb{R}^2$ at points $x^0 = (1,1)$, $x^1 = (1,2)$, $x^2 = (2,1)$. Also, find and draw the corresponding derivatives (affine approximations) of the three level curves det$h^0(x)$, $h^1(x)$, $h^2(x)$.

(b) Give arguments for the existence of a solution, and verify that no solution has $x_1 = 0$ or $x_2 = 0$.

(c) Identify the boundary point $x^*(a)$ that, for a fixed and given, meets the necessary first-order condition $\nabla f(x^*) + \lambda \nabla g(x^*, a) = \mathbf{0}$ for a solution on the boundary.

(d) Give arguments that show (or at least suggest) that the point $x^*(a)$ indeed is the solution.

(e) Find the maximum value $v(a)$ of f over $X(a)$ and draw a diagram of the graph of the function v.

Exercise 4 Maximization (ex. 8.10)

Consider the program $\max_{x \geq 0} f(x,a)$, where the function $f : \mathbb{R}^2_+ \to \mathbb{R}$ is defined by

$$f(x,a) = \frac{100e^x}{e^x + 100} - ax^2$$

(a) Verify (graphically if you want) that this program has at least one solution, for each $a > 0$, but no solution for $a = 0$.

(b) Provide a necessary first-order condition for interior solutions, and argue why there is no boundary solution.

(c) Verify that there exists a critical value of a, say a_0, such that the solution is unique for every $a \neq a_0$,and such that there exists exactly two solutions a when $a = a_0$. Draw diagrams for $a = 2$, 2.345 and 2.5, respectively.

(d) Verify graphically that the solution correspondence takes a jump at $a = a_0$. Use Berge's Maximum Theorem (suitably adapted) to verify that the solution correspondence is hemi-continuous at that point.

(e) Verify graphically or numerically that the value function appears to be continuous. Use Berge's Maximum Theorem (suitably adapted) to verify that the value function is continuous.

Exercise 5 Local uniqueness (ex. 9.3)

Is the solution $x^0 = (1, 1, 1)$ to the following system of non-linear equations locally unique?

$$\begin{cases} x_1^2 + 2x_2^2 + 4x_3^2 = 7 \\ 4x_1x_2^2 + 7x_2x_3 = 11 \\ \log(x_1x_2x_3) = 1 \end{cases}$$

Exercise 6 System of equations (ex. 9.4)

Consider the following system of equations in $x \in \mathbb{R}^3_+$, for a given parameter $a \in \mathbb{R}$:

$$\begin{cases} x_1 + 2\sqrt{x_2}x_3 = 3 \\ x_2x_3 + x_1x_2 = 2 \\ 2x_1x_2x_3 + 4 = a \end{cases}$$

(a) Write the equation system in the form $f(x, a) = \mathbf{0}$, where $f : \mathbb{R}^4_+ \to \mathbb{R}^3$ and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^3$.

(b) For what value a^0 of the parameter a is $x^0 = (1, 1, 1)$ a solution?

(c) Is f continuously differentiable at x^0?

(d) Suppose $a = a^0$. Is the solution $x^0 = (1, 1, 1)$ then locally unique?

(e) Does the equation system define the solution $x \in \mathbb{R}^3$ as an implicitly defined function g of a, $x = g(a)$, for values of a near a^0?