Mathematics I — Problem Set 5

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Due: **October 13**, 2015 at 12.00. Hand in by email or in my mailbox.

Remember to always use clear arguments, using proofs and counter-examples when needed. Please write clearly (computer typed solutions are greatly appreciated) and carefully state definitions and theorems from the Lecture Notes whenever you use them. Collaboration in smaller groups is encouraged, however everyone needs to hand in their own solutions.

Grading is pass/fail. There will be 5 problem sets in total, each having equal weight. A pass will be rewarded as long you show that you understand and have made an honest effort on all questions.

Note the longer deadline! The last two exercises are voluntary!

Exercise 1 Lipschitz continuity (ex. 5.5.2)

For each of the three functions $f : \mathbb{R} \to \mathbb{R}$ defined below, decide if it is (i) continuous, (ii) locally Lipschitz continuous, (iii) globally Lipschitz continuous, (iv) concave, and/or (v) quasi-concave.

(a) $f(x) = e^{-x^2}$

(b) $f(x) = \pi - |x|$

(c) $f(x) = (\max\{0, x\})^{1/2}$

Exercise 2 The Implicit-Function Theorem (ex 9.6)

Consider the continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, a) = x^2 + a^2 - 1$. What does the Implicit-Function Theorem say about solution function, when (a) $a^0 = 0$, when (b) $a^0 = 1/2$, and when (c) $a^0 = 1$? Draw a diagram showing the set where $f(x, a) = 0$.

*If you find any typos, or think that something is unclear, please email me at adam.altmejd@phdstudent.hhs.se. Good luck!

Exercise 3 KKT (ex. 9.12)

Consider the program $\max(x_1 + x_2)$, subject to the constraints $x_1^2 + x_2 \leq 2$ and $x_1 \leq a$ where a is a positive scalar.

(a) Find constraint functions g_1 and g_2 for each of the two constraints, and write the program in such a form that the Kuhn-Tucker Theorem can be used.

(b) Identify the set X^{CQ} of points x that meet the constraint qualification.

(c) Use the Kuhn-Tucker Theorem to identify the solution set for all $a > 0$. (Hint: there are 2 cases)

(d) How does the solution set change when a changes? Is the solution correspondence u.h.c.? Is it l.h.c.?

(e) How does the maximum value change? Is it continuous in a?

Exercise 4 KKT (exam 2013)

Consider the problem $\max_{x \in B} f(x, a)$ where $B = \left\{x \in \mathbb{R}_+^2 : x_1 \leq (2 - x_2)^3\right\}$, and $f(x, a) \equiv ax_1^2 + x_2$ for some scalar $a > 0$.

(a) Does the maximization program have at least one solution? (You may refer to precisely stated theorems.)

(b) Does f have a *gradient* at each point in B? Find the gradient wherever defined.

(c) Formulate the problem in such a way that you can use the Kuhn-Tucker theorem, and state precisely what the theorem says.

(d) Identify the points $x \in B$ that satisfy the *constraint qualification* and, using the Kuhn-Tucker theorem, solve the program for all values of $a > 0$.

Exercise 5 Sequence of averages

Note: This exercise is voluntary.

Show that for every convergent sequence $(s^t)_{t \in \mathbb{N}} \to s$ there exists a sequence of averages

$$\bar{s}^t = \frac{s_1 + s_2 + \ldots + s_t}{t}$$

s.t. $(\bar{s}^t)_{t \in \mathbb{N}} \to s$

Give an example of a sequence that is divergent but has a convergent sequence of averages.

Exercise 6 Multiplicative utility

Note: This exercise is voluntary.

Given the following utility function:

$$u(c) = c$$

Consider a consumer who has such a utility function that is multiplicative over two goods ($c = (c_1, c_2)$), $U : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$U(c) = \begin{cases} u(c_1)u(c_2) & c \in \mathbb{R}^2_+ \\ 0 & \text{otherwise} \end{cases}$$

First try to show that the function is differentiable at $(0, 0)$ using Proposition 41 from the lecture notes. Explain why you cannot use the proposition, and then use the definition to show that the function is in fact differentiable at this point. (Hint: $c_1 c_2 = \|c\|^2 \sin(\alpha)\cos(\alpha)$ with α such that $\tan(\alpha) = c_2/c_1$.) For extra credit, try to show this using directional derivatives as well.